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# ON SOME OPTIMAL DESIGN CRITERIA OF INHOMOGENEOUS ANISOTROPIC BODIES

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Questions of the optimal design of anisotropic inhomogeneous bodies under different optimality conditions are considered. The domain occupied by the body and the conditions on its boundaries are considered given; the structural parameters characterizing the material properties are functions of the coordinates. An existence theorem is proved, as are certain properties of the considered designs.

**1.** An inhomogeneous anisotropic continuous medium characterized by some structural parameters which are functions of the space coordinates, is considered. The medium is a model of real materials of the type of a metal bonded by high-strength monocrystals, fiberglass, etc.; the physical meaning of the structural parameters can be elucidated by turning to specific composites.

Let us say that the body configuration is given if a volume $V$ filled with a continuous medium and the conditions on its boundary surface $\Omega$ are defined. The surface stress resultants are given to the accuracy of some general parameter $t$ (the proportional loading), there are no mass forces, and the possibility of buckling is excluded. The functional relationships connecting the stresses, strains, and structural parameters are called the law of the medium.

The body design will be considered constructed if the functions governing the structural parameters of the medium have been selected in such a manner that the strength conditions at each point, the equilibrium equations, the strain compatibility conditions, and the boundary conditions are satisfied. Appropriate equalities on the surfaces of discontinuity should be satisfied for discontinuous fields.

The problem of designing minimum-weight components (mass, cost of material) results in a typical variational calculus problem, to seek the extremum of the functional which is an additive function of the domains. Known methods of solving such a problem rely essentially on this additivity property.

For a given configuration and law of the medium, the body strength (the global strength) is determined by the maximum value of the loading parameter $t$ which we denote by $p$ ($p = \max t$). The quantity $p$ depends on the selection of functions governing the structural parameters, i.e. is a functional. This functional is not additive since it has no meaning for part of the volume $V$. The problem can be posed of seeking the maximum of the functional as the structural parameters are varied: $p^* = \max p$. Another example of an extremal problem for a nonadditive functional is obtained in constructing a body having the least displacement at a certain point.

Let us consider continuous media which simulate a structurally-anisotropic inhomogeneous material consisting of a binder and a filler (armature) imbedded therein. Let $\rho$ and

$\rho_0$ be the armature and binder densities, $s$ is the content by volume of the armature per unit volume of composite, $v$ is the total volume of the armature in the body. Then $V - v$ and $1 - s$ are the total and relative volumes of the binder, $(\rho - \rho_0)s + \rho_0$ is the mean density of the composite. For given constant $\rho, \rho_0, V$ the armature mass $m$ completely determines the mass of the whole body $M$

$$M = \rho v + \rho_0 (V - v) = (\rho - \rho_0)\, m / \rho + \rho_0 V \qquad \left(m = \rho \int\limits_V s dV\right)$$

For $\rho > \rho_0$ the problem of minimizing the mass of the body reduces to minimizing the mass of the armature. All the quantities introduced correspond to the initial unstrained state of the body; the values for the state of strain will be provided with primes.

Let us examine three optimality criteria for which the following optimal design definitions are taken.

1. An equal-strength (more accurately, equal-strain) design having relative elongations constant over the volume and mutually equal in some three noncoplanar directions for any running value of the external load parameter $t : \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon(t)$, so that for an ultimately loaded body $(t = p)$

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon(p) = \pm \varepsilon^* \tag{1}$$

Here $\varepsilon^*$ is a material constant, $|\varepsilon_i| \leqslant \varepsilon^*$ is a local strength condition $(i = 1,2,3)$.

2. The design of least armature mass for a given strength $p : m_* = \min m\ (p = \text{const})$.

3. The design of maximum strength for a given value of the armature mass $m$: $p^* = \max p\ (m = \text{const})$.

Two characteristic parameters are contained in these definitions: $m$ and $p$; it is desirable that the design, which is optimal in one of the parameters, should be satisfied in the other. If necessary, some combination of these quantities can be introduced, and another definition of an optimal design can be considered.

The purpose herein is to clarify the interrelation between the properties of designs with identical configuration, constructed according to different criteria and having a different internal structure.

Assumptions about the law of the medium will be made as necessary in order that each assertion expressed should be valid for the whole succeeding exposition. The first such assumption is: When a body element is in the state of uniform, multilateral expansion (compression), its behavior is linearly elastic, where the relationship between the first invariant of the stress tensor $\sigma$, the linear strain $\varepsilon$ and the filler content is determined by the equality

$$\sigma = [as + b(1 - s)]\varepsilon \qquad (a,\ b = \text{const}) \tag{2}$$

Theorem 1. For a given body configuration and strength, all designs which are optimal according to criterion 1, and with identical material constants $a, b, \varepsilon^*$ have the same mass.

A substantial coordinate system $\xi_k$ $(k = 1,2,3)$ is introduced for the proof, in which the stress and strain tensor components are denoted by $p^{ij}$ and $\varepsilon_{ij}$, respectively, and the metric tensor components in the initial and running positions by $g_{ij}$ and $g_{ij}'$. The relation between the coordinate vectors $\mathfrak{Э}_k, \mathfrak{Э}_k'$ the systems in the initial and running states, is written as $\mathfrak{Э}_k' = (1 + \varepsilon)\mathfrak{Э}_k$, $(g_{ij}' = (1 + \varepsilon)^2 g_{ij}$, $dV' = (1 + \varepsilon)^3 dV)$ according to criterion 1. Hence $\sigma = p^{ij} g_{ij}'$, and the increment in the strain tensor component during the process of deformation can be written [1]

$$d\varepsilon_{ij} = 0.5d\,(\partial_i'\partial_j' - \partial_i\partial_j) = (1+\varepsilon)\,g_{ij}d\varepsilon = g_{ij}'\,\frac{d\varepsilon}{(1+\varepsilon)}$$

The work of the internal stress resultants during body strain from the initial ($\varepsilon = 0$, $t = 0$) state to the terminal state ($\varepsilon = \varepsilon^*$, $t = p$) is

$$w = \int_{V'} dV' \int_{\varepsilon=0}^{\varepsilon=\varepsilon^*} p^{ij}\,d\varepsilon_{ij} = \int_{V'} dV' \int_0^{\varepsilon^*} \frac{\sigma}{1+\varepsilon}\,d\varepsilon = \int_V dV \int_0^{\varepsilon} \sigma\,(1+\varepsilon)^2\,d\varepsilon =$$

$$= \int_0^{\varepsilon^*} d\varepsilon \int_V [as + b\,(1-s)]\,\varepsilon\,(1+\varepsilon)^2\,dV = [(a-b)\,v + bV]\,\Phi(\varepsilon^*)$$

Here $\Phi(\varepsilon^*)$ is a function obtained as a result of integration. (Henceforth, summation is not carried out over the repeated subscripts).

The deformation of such a design can be considered as a uniform expansion (or compression) relative to some fixed center, hence, the displacement vector of any point can be expressed in terms of the radius-vector of its initial position: $\mathbf{U} = \varepsilon\mathbf{r}$, then $d\mathbf{U} = \mathbf{r}d\varepsilon$ is the increment in the displacement vector during the process of deformation. Writing the intensity of the external surface forces as $t\mathbf{f}\,(\xi_1, \xi_2, \xi_3)$ we obtain an expression for the increment in the work of the external forces on a surface element $dA = t\,(\mathbf{f}, \mathbf{r})\,d\varepsilon d\Omega'$ ($d\Omega' = (1+\varepsilon)^2 d\Omega$). Using the assumption on linear elasticity of the material under multilateral expansion (compression), we find that $\varepsilon = \varepsilon^* t\,/\,p$.

Indeed, the boundary surface changes similarly as the parameter $t$ grows, and the boundary stresses vary in proportion to $t$. If the solution of the problem exists for $t=1$ and defines the stress field $p^{ij}(1)$, then for any $t$ a solution with the field $tp^{ij}(1)$ exists. The exclusion of the possibility of buckling assumes uniqueness of the solution. Therefore, for some fixed design undergoing the mentioned strain, the stress tensor and its first invariant $\sigma$ vary in proportion to the external loading parameter $t$. Using the relation (2), we arrive at the dependence written down for the strain.

The work of the external forces is

$$A = \int_\Omega d\Omega \int_0^p (\mathbf{f}, \mathbf{r})\left(1 + \frac{\varepsilon^* t}{p}\right)^2 \frac{\varepsilon^* t}{p}\,dt$$

Equating the work of the external and internal forces, we determine the filler volume

$$v = [A - bV\Phi(\varepsilon^*)]\,/\,(a-b)\;\;\Phi(\varepsilon^*) \tag{3}$$

Functions dependent on the material constants $a$, $b$, $\varepsilon^*$, the body configuration, and the design strength $p$ enter into the right side of (3). However, the filler volume is independent of any characteristics of the filler distribution over the volume $V$. Any designs of this configuration, with the strength $p$ and constructed from a given material, will have the same filler volume. Considering the density $\rho$ constant, and using the expression written earlier for the mass of the design $M$, we easily see the validity of the remark expressed.

**2.** The further exposition refers to materials whose filler is thin oriented fibers. Let us imagine some three-dimensional curvilinear coordinate system $\xi_i$ with the coordinate vectors $\partial_i$, and let us consider the fibers of the composite being simulated to be parallel to the coordinate lines of this system.

The volume content of fibers in the $i$th direction per unit volume of composite will be denoted by $s_i$, then $s = s_1 + s_2 + s_3$. The quantity $s$ should not exceed the ultimate admissible value $s^* < 1$, determined from the technological considerations. It can be assumed that $s_3 = 0$ for plates and shells produced from such a material.

Assuming the transverse dimension of the fibers arbitrarily small, let us make the transition, customary in solid mechanics, to a continuous, inhomogeneous, anisotropic medium, whose structural parameters $s_1$, $s_2$, $s_3$ are functions of the coordinates $\xi_1$, $\xi_2$, $\xi_3$. By definition, the system $\xi_i$ is substantial.

Let us examine only rational [2] designs which, by definition, possess the following properties:

a) The stress vector $\mathbf{p}^1$ on an area formed by the coordinate vectors $\mathbf{Э}_2'$, $\mathbf{Э}_3'$ is collinear to the vector $\mathbf{Э}_1'$; an analogous assumption is made relative to the stress vectors in the other coordinate areas with a circular permutation of the subscripts;

b) The angles between the coordinate axes of the system $\xi_i$ coupled to the fiber directions are conserved during deformation.

In substance, the assumptions (a) and (b) reduce to the fact that neither shear stress resultants nor shear strains originate between fibers of one family. In this connection, there is no need to determine specifically the relation between these state characteristics in the law of the medium and the strength conditions. The number of governing parameters is thereby cut down substantially; for instance, the body strain at a point is characterized only by the relative fiber elongations $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$.

It is assumed that the binder stiffness is adequate to conserve the composite monolithic and to eliminate the possibility of fiber buckling. Besides the invariant strain characteristics $\varepsilon_i$, invariant quantities governing the state of stress $\sigma_i$ $(i = 1,2,3)$ are also introduced, which can be called the reduced stresses on the coordinate areas.

Let us use the notation $p^i = |\mathbf{p}^i|$ and $\alpha_i$, the angle between the covariant vector $\mathbf{Э}_i'$ and the contravariant vector of the same system $\mathbf{Э}'^i$. Evidently the angle between the coordinate area $\mathbf{Э}_2'$ $\mathbf{Э}_3'$ and its projection on a plane orthogonal to the direction $\mathbf{Э}_1'$ equals $\alpha_1$. By assumption, let $\sigma_i = p^i/\cos\alpha_i$ $(i = 1,2,3)$, then in particular, $\sigma_1$ is the stress resultant on some area formed by the fiber directions $\xi_1$, $\xi_2$ referred to the area of the projection of this area on a plane normal to the direction of the stress resultant.

Using the formulas of Sect. 4 in [1] with a certain difference in the notation, we can write

$$p^i = p^{ii}\sqrt{g_{ii}'/g'^{ii}}, \qquad \cos\alpha_i = (g_{ii}'g'^{ii})^{-1/2}$$

then $\sigma_i = p^{ii}g_{ii}'$ (no summation), and the first invariant of the stress tensor $\sigma$ is represented as $\sigma = \sigma_1 + \sigma_2 + \sigma_3$. The quantities $\sigma_i$ are the principal stresses for an orthogonal coordinate system $\xi_i$ since by virtue of the assumption of rationality of the designs, the principal axes of the stress tensor will coincide with the fiber directions. Hence

$$\sigma_i = p^{ii}, \qquad \varepsilon_i = \sqrt{1 + 2\varepsilon_{ii}/g_{ii}} - 1$$

Let the relationship between the invariant stress and strain characteristics $\sigma_i$, $\varepsilon_i$ and the structural parameters $s_i$, $\alpha_i$ (the law of the medium) be defined by the equalities

$$f_k\,(\sigma_1, \sigma_2, \sigma_3;\ \varepsilon_1, \varepsilon_2, \varepsilon_3;\ s_1, s_2, s_3;\ \alpha_1, \alpha_2, \alpha_3) = 0 \qquad (k = 1,2,3) \tag{4}$$

The following condition is imposed on the function (4): For bounded values of the variables $\sigma_i$, $\varepsilon_i$, $\alpha_i$ satisfying the inequalities $\sigma_i\varepsilon_i \geqslant 0$ and $0 \leqslant \alpha_i \leqslant \pi$ they implicitly determine single-valued bounded positive functions $s_k(\sigma_1, \sigma_2, \sigma_3;\ \varepsilon_1, \varepsilon_2, \varepsilon_3;\ \alpha_1, \alpha_2, \alpha_3)$ such that

$$(\operatorname{sign}\,\sigma_i)\partial s_h / \partial\sigma_i \geqslant \lambda > 0 \qquad (\lambda = \text{const}) \tag{5}$$

The strength condition of the body at a point is $|\,\varepsilon_i\,| \leqslant \varepsilon^*$ $(i = 1,2,3)$.

Let us call a stress field regular if it is statically admissible for a given body configuration, and its isostats (lines of principal stresses) can be taken as the coordinate lines of some curvilinear coordinate system $\xi_i$.

Theorem 2. If multilateral uniform expansion (compression) does not contradict the kinematic conditions for a given body configuration, then any regular stress field $p_0^{kj}$ with principal stresses of the same sign will generate an optimal design according to the criterion 1.

Indeed, the stress field of the desired design can be selected as $p^{kj}=tp_0^{kj}$. The principal lines of stress of this field determines a coordinate system $\xi_i$ giving the directions of the armature fibers, then because of the orthogonality of the principal axes of the stress tensor $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Assuming $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \pm\varepsilon^*$, we find $s_1$, $s_2$, $s_3$ from the law of the medium (4) as functions of the principal stresses $\sigma_1$, $\sigma_2$, $\sigma_3$.

While $t$ is not fixed, we have a family of equal-strength designs. As the parameter $t$ increases, the principal stresses grow from the zero value, and therefore (see (5) ), the bonding intensities should increase also. Assuming the greatest value of the total bonding intensity to reach $s^*$, we obtain an equality to determine the strength of the strongest design from a given family $\max_{V,t} s = s^*$. Here $p$ is the value of $t$ at which this maximum is reached.

In particular, Theorem 2 permits the elementary design of equal-strength membrane shells if the stress fields of the corresponding isotropic shells with principal stresses of the same sign are known.

Henceforth, the body displacements are considered small so that no difference between the states of the boundary surface before and after loading can be noticed.

Theorem 3. The solution of the elasticity theory problem of an isotropic body with regular stress field determines the rational design of an anisotropic body of the same configuration.

In an isotropic body with Poisson's ratio $\nu' = 0$ and still undetermined value of the Young modulus $E'$, let a stress field $\sigma_i(t) = t\sigma_i^\circ$ and a strain field $\varepsilon_i^\circ$ be realized for a value of the loading parameter $t$ such that $\max\{|\varepsilon_1|, |\varepsilon_2|, |\varepsilon_3|\} = \varepsilon^*$. To construct the design of an anisotropic body let us select its stress and strain fields such that they would equal, respectively, in the ultimate loaded state $\sigma_i(p)$, $\varepsilon_i^\circ$ ($p$ is the greatest value of the loading parameter of an anisotropic design). By virtue of Hooke's law for an isotropic body, we can write $\sigma_i(p) = E'\varepsilon_i^\circ$, then we find the function $s_i(E'\varepsilon_i^\circ, \varepsilon^\circ_i)$ $(\alpha_i = 0)$ from (4). Coaxiality of the stress and strain tensors of an elastic isotropic body assures absence of shear stress resultants and shear strains on the coordinate areas of the system $\xi_i$ if it is selected so that the coordinate lines (and the armature fibers) coincide with the isostat lines.

For values of $E'$ for which $s < s^*$, we have a family of rational designs. Selecting the strongest, we assume $\max_{V,E'} s = s^*$, from which we find the value $E' = E_0'$ at which this maximum is realized. The strength is determined from the condition that the stresses in isotropic and anisotropic bodies are equal $p = E_0'\varepsilon_i/\sigma_i^\circ$. The proof can be carried out analogously for $\nu' \neq 0$.

As a rule, Theorem 3 yields a good initial approximation, and by starting from it the design can be improved and its characteristics brought to the optimal by some kind of criterion.

This last exposition refers just to rational designs with two families of armature directions parallel to some plane $Q$. The middle plane of the body occupies a domain $\omega$ on $Q$ bounded by the contour $L$. The external stress resultants acting on the contour are parallel to the plane $Q$. The height of the body is assumed to equal unity.

For the law of the medium (4) $(s_3 = 0)$ under the condition (1) and the additional condition

$$\operatorname{sign} \varepsilon_i \partial s_k / \partial \varepsilon_i \leqslant \mu < 0 \qquad (\mu = \text{const}) \tag{6}$$

the following theorem is valid.

T h e o r e m 4. For a given body configuration and strength, a design which is optimal according to criterion 1 is also optimal according to criterion 2, or differently, an equal-strength design has the least mass.

The proof of this theorem, presented in [2] (Remark 3) for the simplest law of a medium, is extended to this case without difficulty.

3. Let us examine the properties of plane rational designs for the simplest law of a medium. Henceforth, the binder stiffness is considered to be so much less than the armature stiffness that its contribution to the total stiffness can be neglected.

The law of the medium hence has the form

$$\sigma_i = E s_i \varepsilon_i \qquad (i = 1,2) \tag{7}$$

T h e o r e m 5. Each rational design $(m, p)$ generates a family of rational designs with proportionately lower values of the armature mass and the strength $(\mu m, \mu p)$.

Indeed, by having a design characterized by the quantities $m^\circ$, $p^{1\circ}$, $\sigma^\circ(\xi_1, \xi_2)$, $\varepsilon_i^\circ(\xi_1, \xi_2)$, $s_i^\circ(\xi_1, \xi_2)$, $\alpha_i^\circ(\xi_1, \xi_2)$, a new design with the same strain field $\varepsilon_i(\xi_1, \xi_2) = \varepsilon_i^\circ(\xi_1, \xi_2)$ and proportionately diminished fields of stress $\sigma_i(\xi_1, \xi_2) = \mu\sigma_i^\circ(\xi_1, \xi_2)$ and bonding intensity $s_i(\xi_1, \xi_2) = \mu s_i^\circ(\xi_1, \xi_2)$ can be constructed. By virtue of their linearity the equilibrium equations are not violated, the equalities (7) are conserved, the boundary conditions for the stresses will also be satisfied if the parameter $p$ is replaced by $\mu p$. The mass of the armature of such a design will equal $\mu m$.

T h e o r e m 6. For a fixed value of the armature mass $m_0$, the design $(m_0, p_0)$ which is optimal according to criterion 1, is optimal according to criterion 3, or differently, an equal-strength design has the greatest strength for a given mass.

Let us assume the opposite: There exists a design $(m_0, p)$ with the same mass and greater value of the strength than the equal-strength $p > p_0$. In conformity with Theorem 5, this design can be used to construct a new design with proportionately diminished values of the mass and strength. Namely, by taking $\mu = p_0 / p$, we obtain the design $(\mu m_0, p_0)$. However, in conformity with Theorem 4, such a design does not exist (with the same strength as the equal-strength design, but lesser mass).

An imaginary plane with rectangular Cartesian coordinates $m$, $p$ (Fig. 1) can be introduced for a graphic comparison of the properties of diverse designs. Picturing each design by a point on this plane, we can obtain some geometric type results from the assertions expressed.

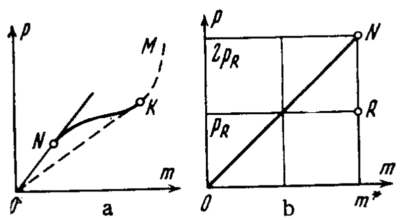

Fig. 1

C o r o l l a r y 1. If there exists an equal-strength design $N$ $(m_0, p_0)$ (see Fig. 1), then a segment of the line $ON$ corresponds to designs which are optimal by all three criteria.

This assertion evidently follows from Theorems

4 – 6 since an equal-strength design generates equal-strength designs.

Besides the quantities $m$ and $p$, their ratio $p/m$, the specific strength of the project, can be considered. All the points on $ON$ correspond to designs of identical specific strength. Applying the reasoning in the proof of Theorem 5, by taking $\mu > 1$, an attempt can be made to construct a design with the same specific strength but with a greater value of the strength $p$; however, this is possible only when the ultimate bonding intensity $s^*$ is not reached at any point of the design $N$.

Considering this possibility to have already been exhausted in the design $N$, i.e. $p$ to be so great that the equality $s = s^*$ has been satisfied at some point of the body, then the following can be formulated.

Corollary 2. The imaginary points of all possible designs are arranged either on the segment $ON$ or below a ray corresponding to this segment.

The absence of points above this ray evidently follows from Theorem 5 and Corollary 1.

According to Corollary 1, the segment $ON$ can be called a section of the optimal design curve. If it is impossible to construct an equal-strength design, then such a curve should correspond to designs, optimal according to Criteria 2 and 3.

From Theorem 5 results –

Corollary 3. The optimal design curve cannot intersect the radius-vector of any of its points twice, i.e. its behavior, pictured by the section $KM$ in Fig. 1a is impossible.

As an illustration, let us consider an annular plate loaded in its plane by a uniform pressure $t$ along the outer contour. There was no body thickness for the computations since the stresses are given on the boundary. It can be considered that a layer of unit thickness is designed. Let $r_0$, $r_1$, $r$ denote the internal, external, and running radii of the plate.

According to the known Lamé formulas, the stress field

$$\sigma_1 = \frac{r_1^2}{r_1^2 - r_0^2}\left(-1 + \frac{r_0^2}{r^2}\right) t \equiv t\sigma_1^\circ \qquad (8)$$
$$\sigma_2 = -\frac{r_1^2}{r_1^2 - r_0^2}\left(1 + \frac{r_0^2}{r^2}\right) t \equiv t\sigma_2^\circ$$

originates in an elastic isotropic ring under an external pressure $t$. The lines of principal stresses are arranged in radial and circumferential directions, which correspond to the subscripts 1 and 2.

Assuming the Poisson ratio zero, let us find the deformation of an isotropic body from the formula $\varepsilon_i = \sigma_i / E'$. Let the strains and stresses of the desired rational design agree with the corresponding quantities for an isotropic body. From the law (7) we find the bonding intensity $s_1 = s_2 = E'/E$. Let us select $E'$ so that the total bonding intensity reaches its greatest value: $s = 2E'/E = s^*$, $E' = Es^*/2$. By virtue of the constancy of $s$ over the whole domain $\omega$, the armature mass will also equal its greatest possible value for a body of given configuration $m_R = m^* = \rho\omega s^*$. The absolute values of the principal strains are bounded by the quantity $\varepsilon^*$, and since the principal stresses are proportional to them, we can write (by putting $t$ equal to the greatest value of $p$)

$$p \max_{\omega} \{|\sigma_1^\circ|, |\sigma_2^\circ|\} = Es^*\varepsilon^*/2 \qquad (9)$$

The domain $\omega$ for the considered body is defined by the inequalities $r_0 \leqslant r \leqslant r_1$ and the circumferential stress $\sigma_2$ reaches the greatest value for $r = r_0$

$$\max_{r_0 \leqslant r < r_1} \{|\sigma_1^\circ|, |\sigma_2^\circ|\} = 2r_1^2 / (r_1^2 - r_0^2)$$

so that it follows from (9) that

$$p = Es^*\varepsilon^* (r_1^2 - r_0^2) / 4r_1^2 \equiv p_R \tag{10}$$

Thus, the bonding directions are selected along the radius and along concentric circles, the bonding intensities are constant: $s_i = s^* / 2$, the strength of the design equals $p_R$, the stresses are determined by means of (8) for $t = p_R$, the strains are $\varepsilon_i = 2\sigma_i/Es^*$. The rational design has been constructed.

Since a regular field of stresses (8) satisfies the condition $\sigma_1\sigma_2 \geqslant 0$, it can be taken as the stress field of an equal-strength design. Following the procedure elucidated in the proof of Theorem 2, let us set $\varepsilon_1 = \varepsilon_2 = -\varepsilon^*$ the $s_i = |\sigma_i| / E\varepsilon^*$ from (7) and

$$s^* \max_{t,\omega} s = \max_{t,\omega} (|\sigma_1| + |\sigma_2|) / E\varepsilon^* \tag{11}$$

or differently

$$p \max_{\omega} (|\sigma_1| + |\sigma_2|) = Es^*\varepsilon^* \tag{12}$$

The stress field under consideration (8) possessed the remarkable property that the sum of its principal stresses is constant, and the equalities (11), (12) are

$$s^* = |\sigma_1 + \sigma_2| / E\varepsilon^*, \qquad p\,|\sigma_1^\circ + \sigma_2^\circ| = Es^*\varepsilon^*$$

so that it turns out that for an equal-strength design

$$p = Es^*\varepsilon^* (r_1^2 - r_0^2) / 2r_1^2 \equiv p_0 = 2p_R, \qquad m_0 = m^*$$

On the image plane $(m, p)$ (Fig. 1b), the point $R$ $(m_R p_R)$ corresponds to a rational design, the point $N$ $(m_0, p_0)$ to an equal-strength design, and the segment $ON$ is the optimal design curve. Since the mass of any design of this configuration cannot exceed $m^*$, no points exist above $N$, i.e. the equal-strength design constructed is absolutely optimal in strength: $p_0 = p^*$.

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